Lecture 16 03/20/2019

Electromagnetic Plane Waves (Cont'd)

Partially Polarized Waves

So far, we have assumed that there is a fixed phase difference between the two components of $\vec{E}$. This is the case for fully polarized waves. In general, however, the phase difference can be random. For a fully unpolarized wave, it is uniformly random over the $(0, 2\pi)$ interval. In fact, even the amplitudes of the $\vec{E}$ components may be mutually random.

These considerations are best exhibited by considering statistical averages of the following bilinear products:

$$\langle |E_x|^2 \rangle, \langle |E_y|^2 \rangle, \langle E_x E_y^* \rangle, \langle E_x^* E_y \rangle$$

These quantities are related to the Stokes parameters defined as:

$$S_0 = |E_x|^2 + |E_y|^2, \quad S_1 = |E_x|^2 - |E_y|^2, \quad S_2 = 2Re(E_x E_y^*), \quad S_3 = 2Im(E_x^* E_y)$$

For a fully polarized wave, we have $S_0^2 = S_1^2 + S_2^2 + S_3^2$. In general, when the wave is partially polarized, we have:

$$S_0^2 \geq S_1^2 + S_2^2 + S_3^2$$

The four bilinear products also define a 2×2 Hermitia matrix called the Coherency matrix J:

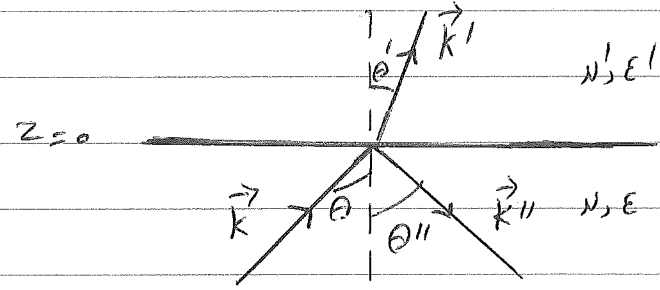
$$J = \begin{bmatrix} \langle |E_x|^2 \rangle & \langle E_x E_y^* \rangle \\ \langle E_x^* E_y \rangle & \langle |E_y|^2 \rangle \end{bmatrix}$$

For a fully polarized wave, we have $\det J = 0$. For a fully unpolarized wave, the off-diagonal entries vanish and $\langle |E_x|^2 \rangle = \langle |E_y|^2 \rangle$, implying that $J \propto \mathbb{1}$. For a partially polarized wave, we have $|\langle E_x^* E_y \rangle|^2 < \langle |E_x|^2 \rangle \langle |E_y|^2 \rangle$, and hence $\det J > 0$.

Wave Propagation in Inhomogeneous Media

The simplest example of an inhomogeneous medium is two semi-infinite homogeneous media with a plane interface. We now discuss the reflection and refraction of an incident plane wave on such an interface.

$n' = \sqrt{\frac{\mu' \epsilon'}{\mu_0 \epsilon_0}}$



θ: angle of incidence
θ': angle of refraction
θ'': angle of reflection

$n = \sqrt{\frac{\mu \epsilon}{\mu_0 \epsilon_0}}$

$\vec{E} = \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$, $\vec{B} = \frac{\vec{k} \times \vec{E}}{\omega} = \frac{n}{c} \hat{k} \times \vec{E}$

$\vec{E}' = \vec{E}_0' \, e^{i(\vec{k}'\cdot\vec{x} - \omega t)}$, $\vec{B}' = \frac{\vec{k}' \times \vec{E}'}{\omega} = \frac{n'}{c} \hat{k}' \times \vec{E}'$

$\vec{E}'' = \vec{E}_0'' \, e^{i(\vec{k}''\cdot\vec{x} - \omega t)}$, $\vec{B}'' = \frac{\vec{k}'' \times \vec{E}''}{\omega} = \frac{n}{c} \hat{k}'' \times \vec{E}''$

Since the fields must satisfy the boundary conditions at the interface

everywhere, we must have:

$$e^{i \vec{k}_{\parallel} \cdot \vec{x}_{\parallel}} = e^{i \vec{k}'_{\parallel} \cdot \vec{x}_{\parallel}} = e^{i \vec{k}''_{\parallel} \cdot \vec{x}_{\parallel}}$$

Here, $\parallel$ indicates parallel to the interface. This means that the vectors

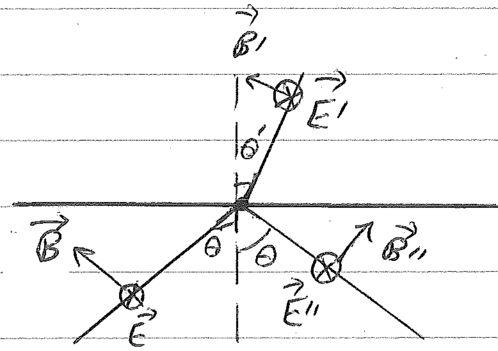
$\vec{k}, \vec{k}', \vec{k}''$ are coplanar (1st Snell's law), and that:

$k \sin\theta = k' \sin\theta' = k'' \sin\theta'' \Rightarrow \boxed{\theta = \theta'', \; n \sin\theta = n' \sin\theta'}$ (2nd Snell's law)

We now impose the boundary conditions at the interface to obtain

the relations between the refracted and reflected waves and the incident wave. For simplicity, we consider two specific orthogonal linear polarizations:

(a) $\vec{E}$ perpendicular to the plane of incidence ("s polarization").



* Continuity of E_t : $\boxed{E_0' = E_0 + E_0''}$

* Continuity of B_n: $B_0' \sin\theta' = B_0 \sin\theta + B_0'' \sin\theta \Rightarrow \frac{n'}{c} E_0' \sin\theta' = \frac{n}{c} E_0 \sin\theta + \frac{n}{c} E_0'' \sin\theta \Rightarrow E_0' n' \sin\theta' = (E_0 + E_0'') n \sin\theta \underset{\substack{\text{2nd Snell's} \\ \text{law}}}{\Longrightarrow} E_0' = E_0 + E_0''$

We see that this is the same relation as that from continuity of E_t.

* Continuity of D_n: trivial since $D_n = D_n' = D_n'' = 0$ for "s polarization".

* Continuity of H_t: $H_0' \cos\theta' = H_0 \cos\theta - H_0'' \cos\theta \Rightarrow \frac{1}{\mu'} n' E_0' \cos\theta' = \frac{1}{\mu} n E_0 \cos\theta - \frac{1}{\mu} n E_0'' \cos\theta \Rightarrow \boxed{E_0' \frac{n' \cos\theta' / \mu'}{n \cos\theta / \mu} = E_0 - E_0''}$

The continuity of D_n and H_t is due to the fact that there are no free charges or currents at the interface. We note that the four boundary conditions yield two independent relations (which are boxed). This is the mathematical reason why only two waves are generated by the incident wave.

We then find:

$$\boxed{r_s \equiv \frac{E_0''}{E_0} = \frac{n\cos\theta - \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta}}{n\cos\theta + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta}}, \quad t_s \equiv \frac{E_0'}{E_0} = \frac{2n\cos\theta}{n\cos\theta + \frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta}}}$$

Here, "r_s" and "t_s" are called the "amplitude reflection coefficient" and the "amplitude transmission coefficient" respectively.

We can also define the "power reflection coefficient" and the "power transmission coefficient" as follows:

$$R_s \equiv \frac{\cos\theta\, S''}{\cos\theta\, S} = \frac{\frac{1}{2}\mathrm{Re}(\vec{E}''\times\vec{H}''^*)\cdot\hat{k}''}{\frac{1}{2}\mathrm{Re}(\vec{E}\times\vec{H}^*)\cdot\hat{k}}$$

Time averaged

$$T_s \equiv \frac{\cos\theta'\, S'}{\cos\theta\, S} = \frac{\cos\theta'}{\cos\theta}\,\frac{\frac{1}{2}\mathrm{Re}(\vec{E}'\times\vec{H}'^*)\cdot\hat{k}'}{\frac{1}{2}\mathrm{Re}(\vec{E}\times\vec{H}^*)\cdot\hat{k}}$$

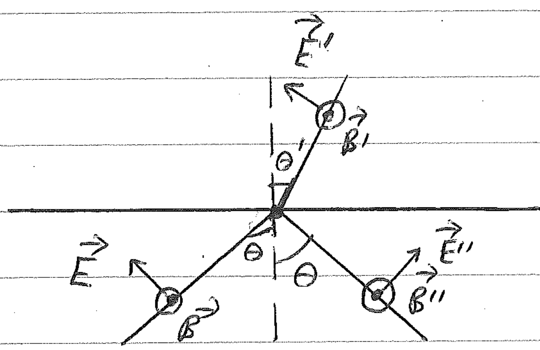
The $\cos\theta$ and $\cos\theta'$ appear since a unit area of the interface corresponds to a cross-sectional are equat to $\cos\theta$ (in the incident and reflected waves) and $\cos\theta'$ (in the refracted wave).

We then find:

$$\boxed{R_s = \frac{\left(\frac{n\cos\theta}{\mu} - \frac{n'\cos\theta'}{\mu'}\right)^2}{\left(\frac{n\cos\theta}{\mu} + \frac{n'\cos\theta'}{\mu'}\right)^2}, \quad T_s = \frac{\frac{4nn'\cos\theta\cos\theta'}{\mu\mu'}}{\left(\frac{n\cos\theta}{\mu} + \frac{n'\cos\theta'}{\mu'}\right)^2}}$$

It is seen that $R_s + T_s = 1$ (as expected).

(b) $\vec{E}$ parallel to the plane of incidence ("p polarization").



Repeating the same steps as in the case of "s polarization", we find:

$$\boxed{r_p \equiv \frac{E_0''}{E_0} = \frac{\frac{\mu}{\mu'} n'^2\cos\theta - n\sqrt{n'^2 - n^2\sin^2\theta}}{\frac{\mu}{\mu'} n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}}, \quad t_p \equiv \frac{2nn'\cos\theta}{\frac{\mu}{\mu'} n'^2\cos\theta + n\sqrt{n'^2 - n^2\sin^2\theta}}}$$

For simplicity, let us assume that $\mu = \mu'$. Then, we see that:

$\theta = \tan^{-1}\left(\frac{n'}{n}\right) \Rightarrow \cos\theta = \frac{n}{\sqrt{n'^2+n^2}}$, $\sin\theta = \frac{n'}{\sqrt{n^2+n'^2}} \Rightarrow E_0'' = 0$

Therefore at the "Brewster angle" $\theta_B \equiv \tan^{-1}\left(\frac{n'}{n}\right)$, there is no reflected wave for "p polarization". This can be used to polariza an unpolarized wave by reflection. We note that $\theta + \theta' = \frac{\pi}{2}$ at the Brewster angle, i.e., the reflected and transmitted directions are perpendicular.

For $\theta = 0$ (normal incidence), both polarizations are parallel to the interface. In this case, we have $t_s = t_p$ and $r_s = -r_p$. The minus sign in the latter expression is due to the fact that E_0 and E_0'' are parallel for "s polarization", while they are anti-parallel for "p polarization", at $\theta = \frac{\pi}{2}$ as seen in the respective figures.